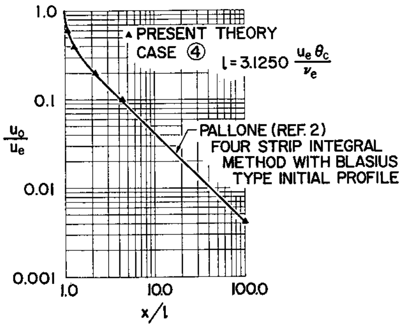

Fig. 2 Velocity decay along axis of incompressible axisymmetric wake

In Fig. 2, a comparison for an axisymmetric wake is made between the present theory and the four-strip integral method (Fig. 4 of Ref. 2). The solution presented in Ref. 2 is plotted in terms of x/l, where l is a constant characteristic length. In Ref. 2 a relation exists between l and the drag of the body, and therefore $(u_e\theta_c/\nu_e)$ is not given. However, the relationship can be calculated from the asymptotic solution [Eq. (8c)] $x = (u_e\theta_c/\nu_e)\varkappa(1 - \bar{u}_0)^{-1}$ by taking from Fig. 4 of Ref. 2 asymptotic values of x and $\bar{u}_0$, namely, $x = 100$ when $1 - \bar{u}_0 = 0.004$. This yields $l = 2.5\varkappa(u_e\theta_c/\nu_e)$. The present theory then is extrapolated upstream by using Eq. (8a).

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Classical Analog of the Photoelectric Effect

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BASED on the analyses of Grad and Lees, the rarefied gas field recently has been shown to possess the unique characteristic that small plane longitudinal and transverse disturbances can propagate within the field.[1–6] Small momentum and energy disturbances produced by impulsive boundary motions cannot be confined to regions near the boundary and can propagate over very large distances in the field before being damped by collisions. This disturbance propagation occurs when the relaxation time is very large and the particle density is very small ($<10^{10}$ particles per unit volume).

If an infinite plate undergoes a small impulsive motion $U(U \ll c)$ in a direction normal to its surface, the disturbance motion produced in a rarefied gas field initially in equilibrium can be described by the linearized one-dimensional longitudinal propagation equations previously derived.[4] If specular reflection does not occur at the boundary and the plate is in thermal equilibrium with the field at temperature T_0, the field particles that collide with the plate can be assumed to be absorbed and re-emitted at equilibrium conditions (p_0,T_0) with a velocity U(Fig. 1). During the impulsive plate motion, the average mass flow $\rho_0\bar{c}$ approaching the plate, boundary A, where $\bar{c}$ is an average random particle velocity, is increased upon re-emission to $\rho_0(\bar{c} + U)$. The characteristic propagating quantities at boundary A become $P_{1\pm} = \mp 0.42U/c_0$, $P_{2\pm} = \pm 1.66U/c_0$.

When these disturbances impinge upon a stationary boundary B, also at temperature T_0, the slow and fast characteristics yield an average disturbance pressure $p = 1.29U/c_0$ at point 1 and $p = 1.61U/c_0$ at point 2. The pressure produced by the longitudinal disturbance is directly proportional to the disturbance velocity. If the disturbances impinge on a free body of mass M_0 and sufficient collisions occur to impart a velocity U (the maximum disturbance velocity that can be imparted, since $p = p_0$ when $u_w = U$), the free body can acquire a maximum momentum $M_p = M_0 U$ and a maximum energy $E_0 = \frac{1}{2}M_0 U^2$.

Physically, the process of momentum and energy transfer occurs by individual field particle collisions. Momentum and energy are imparted to the field particles at the boundary A, and the field particles transfer momentum and energy directly to the body B. The transfer process can be idealized by assuming that the field particle of mass m_0 possesses, after the disturbance motion is imparted to the field particle at the boundary, an average velocity $(\bar{c} + U)$, an average energy $\frac{1}{2}m_0(\bar{c} + U)^2$, and an average momentum $m_0(\bar{c} + U)$. If n_0 collisions are required to produce a maximum velocity U of the free body of mass M_0, then the maximum momentum M_p and maximum energy E_0 transferred are given by

$$M_p = n_0[m_0(\bar{c} + U) - m_0\bar{c}] = n_0m_0 U = M_0 U \qquad \text{(1a)}$$

$$E_0 = \alpha n_0[\tfrac{1}{2}m_0(\bar{c} + U)^2 - \tfrac{1}{2}m_0\bar{c}^2] \approx \alpha n_0 m_0 \bar{c} U = \tfrac{1}{2}M_0 U^2 \qquad \text{(1b)}$$

where α is the relative efficiency of energy transfer, $2\alpha = U/c$. On the average, n_0 collisions will occur in a direction opposite to the motion by particles with an average velocity $\bar{c}$.

The momentum and energy transferred are directly proportional to the amplitude of the disturbance motion. If the impulsive boundary motion of duration t_0 occurs over a fixed distance r_0, $U = r_0/t_0$ and the maximum energy E_0 transferred by the collisions is inversely proportional to the duration, i.e., $E_0 \sim 1/t_0$, or directly proportional to a frequency factor $\omega_0 = 1/t_0$.

An increase in the field density or field intensity, i.e., an increase in the number of boundary collisions per unit time, will not alter the maximum value E_0. A change in the field intensity only will alter the time interval $\Delta t \leqslant t_0$ in which a given value of energy can be imparted to the body, i.e., an uncertainty relation can be defined by

$$\Delta E \ \Delta t \leqslant E_0 \, t_0 = \text{const}$$

which is related to the statistical collision process for the acquisition of the small excess energy $\Delta E \leqslant E_0$.

This process of energy transfer in a rarefied gas field is remarkably similar to the energy transfer process described by the photoelectric effect. If the energy transferred by n_0 field

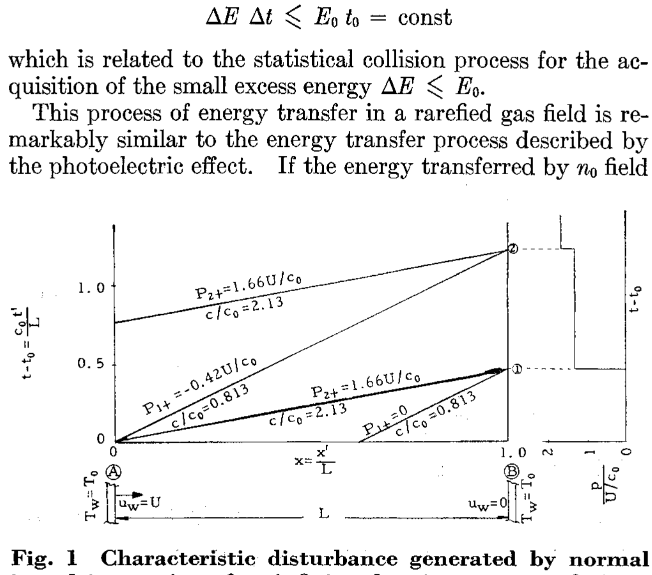

Fig. 1 Characteristic disturbance generated by normal impulsive motion of an infinite plate in a very rarefied gas field

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particle collisions is identified with the energy transferred by a single imaginary field particle of mass $\mu_0 = n_0 m_0$, then

$$E_0 = \alpha\mu_0\bar{c}r_0\omega_0 = \beta_0\omega_0 \qquad \beta_0 = \alpha\mu_0\bar{c}r_0 \tag{2}$$

where β_0 is a constant of dimensions erg-sec in cgs units. This "classical photon" of mass μ_0 will appear to possess an excess energy $E_0' = n_0 m_0 \bar{c} U$ above the average field energy and an excess momentum $M_p = E_0'/\bar{c}$.

The constant β_0 appears to be a classical analog of Planck's constant h in the expression for the energy transferred in the photoelectric effect, $E = h\nu$. Since h can be related to the radius of the first Bohr orbit $a_0 = h^2/4\pi^2 M_e e^2$ and the fine structure constant $\alpha = v/c = 2\pi e^2/ch$,[7] h can be written in the form $h/2\pi = \alpha M_e c a_0$, where e is the electron charge, M_e the electron mass, v the electron velocity in the first Bohr orbit, and c the velocity of light. From Eq. (1a), $n_0 m_0 = M_0$ and β_0 may be written as $\alpha M_0 \bar{c} r_0$. On comparing this expression with that for $h/2\pi$, the interpretation of $h/2\pi$ as a constant that characterizes the electromagnetic field in the same manner that β_0 characterizes the rarefied gas field is suggested immediately.

During the collision process in the rarefied gas field, if a pressure p_i is not generated and a free body acquires a velocity $u_i < U$, the momentum P_i acquired is given by

$$P_i = M_0 u_i = \frac{n_i}{n_0}\mu_0 U = \frac{n_i}{n_0}\frac{E_0}{\alpha\bar{c}} = \frac{n_i}{n_0}\left(\frac{\beta_0\omega_0}{\alpha\bar{c}}\right), \quad n_i \leqslant n_0 \tag{3}$$

and the energy E_i acquired is given by

$$E_i = \frac{P_i^2}{2M_0} = \frac{1}{2}M_0 u_i^2 = \left(\frac{n_i}{n_0}\right)^2 E_0 = \left(\frac{n_i}{n_0}\right)^2 \beta_0\omega_0,$$

$$\frac{u_i}{U} = \frac{n_i}{n_0} \tag{4}$$

Here, n_i is the number of collisions required to produce the velocity u_i, and it is assumed that each collision transfers equal values of momentum and energy.

On substituting for P_i from Eq. (3) in Eq. (4) and defining the state ω_i by $\omega_i = (n_i/n_0)^2\omega_0$, the following relation can be obtained:

$$\frac{\omega_i}{2\pi\bar{c}} = \frac{\alpha'}{2\pi r_0}\left(\frac{n_i}{n_0}\right)^2 = R_0\frac{1}{(n_0/n_i)^2},$$

$$R_0 = \frac{\alpha'}{2\pi r_0}, \qquad \alpha' = 2\alpha \tag{5}$$

The difference in energy between any two states i and j, ν_{ij}, can be written in the form

$$\nu_{ij} = \frac{\omega_i}{2\pi\bar{c}} - \frac{\omega_j}{2\pi\bar{c}} = R_0\left[\frac{1}{(n_0/n_i)^2} - \frac{1}{(n_0/n_j)^2}\right] \tag{6}$$

The difference ν_{0j} between the maximum energy state E_0, $n_i = n_0$, and the jth energy state is

$$\nu_{0j} = R_0\left(1 - \frac{1}{n^2}\right) \qquad n^2 = \left(\frac{n_0}{n_j}\right)^2, \quad 1 < n \leqslant \infty \tag{7}$$

ν_{0j} is identical in form with the Lyman series for the hydrogen atom[7]:

$$\nu_n = R\left(\frac{1}{1^2} - \frac{1}{n_2^2}\right), \qquad n_2 = 2, 3, 4 \ldots \tag{8}$$

if n is an integer and R is the Rydberg constant $2\pi^2 M_e e^4/ch^3$, where M_e is the electron mass, e the electron charge, c the velocity of light, and h Planck's constant. Since the fine structure constant is defined by $\alpha = 2\pi e^2/ch$ and h may be written in the form $h/2\pi = \alpha M_e c a_0$, where a_0 is the radius of the first Bohr orbit, the Rydberg constant also may be written in the form $R = \alpha/2\pi(2a_0) = \alpha/2\pi d_0$.

Consequently, the constant R_0 may be considered to be a classical analog of the Rydberg constant R. The difference in energy between any intermediate level j and the maximum energy level is determined by a "quantum" number n that is a collision ratio, Eq. (7).

The free particle, as it acquires its "quantized" energy in the discrete "jumps" defined by Eq. (6), will in turn produce longitudinal disturbances in the rarefied gas field which possess similar energy values (Fig. 2). The discrete nature of the energy spectrum only occurs when the ratio $m_0/M_0 \approx 1$ and the collision can produce a motion of the free body. If $m_0/M_0 \ll 1$, a pressure also will be produced at the surface of the free body. The limits for the discrete or "quantized" motion and the "classical" continuous motion (the motion produced by a pressure Δp only) occur for $m_0/M_0 \approx 1$ and $m_0/M_0 \approx 0$, respectively. These are the same limits for which $\alpha =$ const and $\alpha = 0$ and $\beta_0 =$ const and $\beta_0 = 0$ and are identical with the limits usually employed to define the transition between quantum mechanics and classical mechanics, i.e., $h =$ const and $h = 0$.

The excess momentum and energy of the "classical photon" are $n_0 m_0 U = E_0'/\bar{c}$ and $n_0 m_0 \bar{c} U = E_0'$, respectively. Hence $n_0 m_0 U = E_0'/\bar{c} = \beta_0\omega_0/\alpha\bar{c}$ and a wavelength λ may be associated with the "classical photon" motion or propagation,

$$\lambda = \frac{2\pi\bar{c}}{\omega} = \frac{\bar{c}}{\omega_0} \qquad \frac{\omega_0}{\bar{c}} = \frac{\alpha\mu_0 U}{\beta_0} \tag{9}$$

The longitudinal characteristic disturbance quantities $P_\pm$ are linear, and each component of $P_\pm$ satisfies the propagation equations.[5] Consequently, the propagation equations satisfied by small velocity disturbances may be written, for a single idealized propagation velocity $\bar{c}$, in the form

$$\left[\frac{\partial}{\partial t} \pm \bar{c}\frac{\partial}{\partial x}\right] u\,(x,t) = 0; \quad \left[\frac{\partial^2}{\partial t^2} - \bar{c}^2\frac{\partial^2}{\partial x^2}\right] u(x,t) = 0 \tag{10}$$

Equation (10) also may be written in the form

$$\left[\frac{\partial^2}{\partial t^2} - \bar{c}^2\frac{\partial^2}{\partial x^2}\right] E = 0; \quad E = n_0 m_0 \bar{c} u(x,t) = \mu_0 \bar{c} u(x,t) \tag{11}$$

since $n_0 m_0 \bar{c}$ is a constant. Consequently, the excess energy associated with the "classical photon" can be assumed to propagate with an average velocity $\bar{c}$.

The "classical photon" energy E_0' can be identified with the energy E_0 acquired by the particle of mass M_0. Since a wavelength and frequency can be associated with the "classical photon," the same wavelength and frequency can be associated with the particle M_0:

$$\frac{\omega_0}{\bar{c}} = \frac{\alpha\mu_0 U}{\beta_0} = \frac{\alpha M_0 U}{\beta_0}, \qquad \mu_0 = M_0 \tag{12}$$

The particle with mass M_0, however, moves with an approximate velocity $\bar{u} = U/2$ in the field (Fig. 2) and consequently will satisfy equations of the form

$$\left[\frac{\partial}{\partial t} \pm \bar{u}\frac{\partial}{\partial x}\right] \phi\,(x,t) = 0; \quad \left[\frac{\partial^2}{\partial t^2} - \bar{u}^2\frac{\partial^2}{\partial x^2}\right] \phi(x,t) = 0 \tag{13}$$

On making the assumption that the free body possesses a particle solution ϕ of the form

$$\phi(x,t) = \psi(x)e^{-i\omega t} = \psi(x)e^{-2\pi i\omega_0 t}$$

Eq. (13) can be written in the form

$$\nabla^2\psi + \frac{16\pi^2\alpha^2 M_0^2\bar{c}^2 U^2}{\beta_0^2 U^2}\psi = 0 \tag{14a}$$

or

$$\nabla^2\psi + \frac{4\pi^2 M_0^2 U^2}{\beta_0^2}\psi = 0 \tag{14b}$$

The one-dimensional free particle energy distribution func-

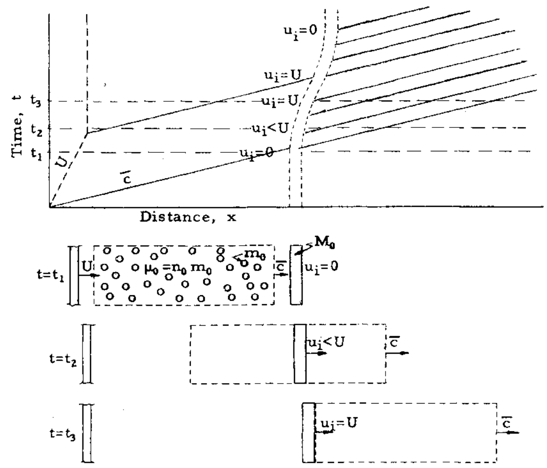

Fig. 2 Idealized collision process in rarefied gas field; $U << \bar{c}$, $\mu_0 = M_0$

tion $\psi(x)$ consequently is determined by the solution of the "wave" equation:

$$\nabla^2\psi + (8\pi^2 M_0 E_0/\beta_0^2)\psi = 0 \qquad (15)$$

The free body satisfies a corresponding momentum equation of the form

$$\frac{1}{2\pi i}\frac{\partial\psi}{\partial x} \pm \frac{2\omega_0}{U}\psi = 0 \quad \text{or} \quad \frac{1}{2\pi i}\frac{\partial\psi}{\partial x} \pm \frac{M_0 U}{\beta_0}\psi = 0 \qquad (16)$$

If the free body possesses a potential energy V, the total energy may be written as $E_T = \frac{1}{2}M_0U^2 + V$. Hence, $M_0U = (1/\alpha\bar{c})(E_T - V)$, and Eq. (16) can be written in the alternate form

$$\frac{\beta_0\bar{c}}{2\pi i}\frac{\partial\psi}{\partial x} \pm \frac{1}{\alpha}(E_T - V)\psi = 0 \qquad (17)$$

Equation (15) is a one-dimensional classical analog of the Schrodinger equation, which also may be written in the form

$$\nabla^2\psi + (8\pi^2 M_0/\beta_0^2)(E_T - V)\psi = 0 \qquad (18)$$

if the particle possesses a potential energy. Equation (17) is a classical analog of the free particle momentum equation and, if V is identified with an energy $M_0\bar{c}^2$, is a classical analog of the Dirac relativistic wave equation for a free particle.[8,9]

Further details of the characteristics of the idealized monatomic rarefied gas field described by approximate solutions of the Boltzmann equation will be described in forthcoming papers. The observations that transverse disturbances can propagate in such fields,[3,5] that the vector forms of the equations are very similar to the Maxwell equations,[6] and that classical analogs of the equations of quantum mechanics may exist suggest that the significance of these linearized rarefied gas field approximate solutions is much greater than previously has been appreciated.

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Invariant Two-Body Velocity Components and the Hodograph

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Nomenclature

e = eccentricity of conic figure
F = attracting focus of conic figure
h = specific angular momentum
r = radial distance between gravitational centers of orbital and celestial bodies
V = orbital velocity
V_h = horizontal invariant velocity component
V_p = invariant velocity component normal to the lines of apsides, $\mathbf{V} = \mathbf{V}_h + \mathbf{V}_p$
V_r = radial velocity component
V_θ = horizontal velocity component, $\mathbf{V} = \mathbf{V}_r + \mathbf{V}_\theta$
V_x = velocity component parallel to the apsidal line
V_y = velocity component normal to the apsidal line, $\mathbf{V} = \mathbf{V}_v + \mathbf{V}_y$
β = flight path angle
θ = true anomaly
μ = gravitational parameter of the two-body system

In a note published recently,[1] Cronin and Schwartz draw the readers' attention to the "little used property of two-body orbital motion, that the velocity vector at any point can be resolved into two components of invariant magnitude." They write also that "no inferences seem to have been made that there is any practical application which takes advantage of the invariance of these two velocity components."

In a series of articles presented recently,[2,4,5] Altman and Pistiner analyzed different two-body problems by use of the special hodograph. At the 2nd International Symposium on Rockets and Astronautics in Tokyo in 1960, Fang-Toh-Sun introduced also a special hodograph for the solution of orbital problems.[3,6]

The purpose of this note is to show the application of the invariance of two velocity components to the derivation of both the classical (Hamilton) and the special hodograph.

Consider a two-body conic trajectory with focus F. The velocity V can be resolved (Fig. 1): 1) into two components parallel and normal to the line of apsides, $\mathbf{V} = \mathbf{V}_x + \mathbf{V}_y$; 2) into two components along and normal to the radius vector from the focus, $\mathbf{V} = \mathbf{V}_r + \mathbf{V}_\theta$; and 3) into two components of invariant magnitude V_h normal to the radiusvector and V_p normal to the line of apsides, $\mathbf{V} = \mathbf{V}_h + \mathbf{V}_p$.

Here, according to Ref. 1, $V_h = \mu/h$, a constant and everywhere normal to the radius vector, and $V_p = (\mu/h)e$, a constant and everywhere normal to the line of apsides.

From the triangle of velocities ΔABS, one has the equation of classical (Hamilton) hodograph:

$$V_x^2 + (V_y - V_p)^2 = V_h^2$$

In the plane V_x, V_y the hodograph is a circle; its center is placed on the V_y axis at a distance V_p from the origin, and its radius is equal to V_h (Fig. 2).

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